An Approximate Solution of the Master Equation with the Dissipator being a Set of Projectors

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Abstract

In this paper we consider a quantum open system and treat the master equation with some restricted dissipator which consists of a set of projection operators (projectors). The exact solution is given under the commutable approximation (in our terminology). This is the first step for constructing a general solution.

In this paper we revisit dynamics of a quantum open system. First of all we explain our purpose in a short manner. See [1] as a general introduction to this subject. We consider a quantum open system S coupled to the environment E. Then the total system $S+E$ is described by the Hamiltonian

$$H_{S+E} = H_S \otimes \mathbf{1}_E + \mathbf{1}_S \otimes H_E + H_I$$

where H_S, H_E are respectively the Hamiltonians of the system and environment, and H_I is the Hamiltonian of the interaction.

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Then under several assumptions (see [1]) the reduced dynamics of the system (which is not unitary !) is given by the Master Equation

$$\frac{\partial}{\partial t}\rho = -i[H_S, \rho] - \mathcal{D}(\rho) \tag{1}$$

with the dissipator being the usual Lindblad form

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_{\{j\}}\left(A_j^\dagger A_j\rho + \rho A_j^\dagger A_j - 2A_j\rho A_j^\dagger\right). \tag{2}$$

Here $\rho \equiv \rho(t)$ is the density operator (matrix) of the system.

It is not easy to solve the equation (1) with the dissipator (2), so we make a simple and convenient assumption. Namely, the generators $\{A_j\}$ are given by $A_j = \sqrt{\lambda_j}P_j$ with projectors $\{P_j\}$; $P_j^\dagger = P_j,\ P_j^2 = P_j,\ P_jP_k = \delta_{jk}P_k$. Note that we don't assume the rank P_j =1 (extended models). Then the dissipator becomes

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_{\{j\}}\lambda_j\left(P_j\rho + \rho P_j - 2P_j\rho P_j\right) \tag{3}$$

where $\{\lambda_j\}$ are decoherence parameters to determine the strength of the interaction. See [2], [3] (in [3] there is a very compact description on this subject). It is interesting to rewrite (3) as

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_{\{j\}}\lambda_j\left\{P_j\rho(\mathbf{1} - P_j) + (\mathbf{1} - P_j)\rho P_j\right\} \equiv \frac{1}{2}\sum_{\{j\}}\lambda_j\left(P_j\rho Q_j + Q_j\rho P_j\right). \tag{4}$$

Note that $\{Q_j\}$ are also projectors satisfying $P_jQ_j = Q_jP_j = 0$ for $j \in \{j\}$.

As a result we have only to solve the equation

$$\frac{\partial}{\partial t}\rho = -i(H\rho - \rho H) - \frac{1}{2}\sum_{\{j\}}\lambda_j\left(P_j\rho Q_j + Q_j\rho P_j\right) \tag{5}$$

where we have set $H = H_S$ for simplicity.

In order to attack the equation (5) let us make some mathematical preliminaries. For a matrix $X = (x_{ij}) \in M(n; \mathbf{C})$ we correspond to the vector $\widehat{X} \in \mathbf{C}^{n^2}$ as

$$X = (x_{ij}) \ \longrightarrow\ \widehat{X} = (x_{11}, x_{12}, \cdots, x_{1n}, \cdots\cdots, x_{n1}, x_{n2}, \cdots, x_{nn})^T \tag{6}$$

where T means the transpose. Then the following formula is well–known

$$\widehat{AXB} = (A \otimes B^T)\widehat{X} \tag{7}$$

for $A, B, X \in M(n;\mathbf{C})$. Since the proof is easy we leave it to readers.

By use of the formula the equation (5) can be rewritten as

$$\frac{\partial}{\partial t}\widehat{\rho} = \left\{ -i(H \otimes \mathbf{1} - \mathbf{1} \otimes H^T) - \frac{1}{2}\sum_{\{j\}} \lambda_j \left(P_j \otimes Q_j^T + Q_j \otimes P_j^T \right) \right\} \widehat{\rho}, \tag{8}$$

therefore the formal solution is given by

$$\widehat{\rho}(t) = \exp\left\{ -it(H \otimes \mathbf{1} - \mathbf{1} \otimes H^T) - t\sum_{\{j\}} (\lambda_j/2) \left(P_j \otimes Q_j^T + Q_j \otimes P_j^T \right) \right\} \widehat{\rho}(0). \tag{9}$$

To calculate $\exp(\cdots)$ explicitly is (almost) impossible, so we must appeal to some approximation method. For that let us remind the Baker–Campbell–Hausdorff (B-C-H) formula. For $A, B \in M(n;\mathbf{C})$ we want to decompose as

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{I(A,B)}\mathrm{e}^{B}. \tag{10}$$

The "interaction" term $I(A,B)$ is given by

$$I(A,B) = -\frac{1}{2}[A,B] + \frac{1}{6}\left\{ [[A,B],B] + [A,[A,B]] \right\} + \cdots . \tag{11}$$

The proof is easy. In fact, $\mathrm{e}^{I(A,B)} = \mathrm{e}^{-A}\mathrm{e}^{A+B}\mathrm{e}^{-B}$ by (10) and we have only to apply the B-C-H formula ([4] and see also [5] as an interesting topic)

$$\mathrm{e}^{X}\mathrm{e}^{Y} = \mathrm{e}^{X+Y+(1/2)[X,Y]+(1/12)\{[[X,Y],Y]+[X,[X,Y]]\}+\cdots} \quad \text{for } X, Y \in M(n;\mathbf{C})$$

two times.

For

$$A = -it(H \otimes \mathbf{1} - \mathbf{1} \otimes H^T), \quad B = -t\sum_{\{j\}} (\lambda_j/2) \left(P_j \otimes Q_j^T + Q_j \otimes P_j^T \right)$$

there is no method to calculate $\mathrm{e}^{I(A,B)}$ explicitly as far as we know. Therefore we ignore this term, namely let us call it the "commutable approximation".

Under the commutable approximation we have only to calculate

$$
\begin{aligned}
\widehat{\rho}(t) &\approx \exp\left\{-it(H\otimes\mathbf{1}-\mathbf{1}\otimes H^T)\right\}\exp\left\{-t\sum_{\{j\}}(\lambda_j/2)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\}\widehat{\rho}(0)\\
&=\left(\mathrm{e}^{-itH}\otimes\mathrm{e}^{itH^T}\right)\exp\left\{-t\sum_{\{j\}}(\lambda_j/2)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\}\widehat{\rho}(0)\\
&=\left(\mathrm{e}^{-itH}\otimes\left(\mathrm{e}^{itH}\right)^T\right)\exp\left\{-t\sum_{\{j\}}(\lambda_j/2)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\}\widehat{\rho}(0). \qquad (12)
\end{aligned}
$$

Next let us calculate the second term in (12), which is not so difficult as follows.

$$
\begin{aligned}
(\sharp) &\equiv \exp\left\{-t\sum_{\{j\}}(\lambda_j/2)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\}\\
&=\prod_{\{j\}}\exp\left\{(-\lambda_j t/2)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\}\\
&=\prod_{\{j\}}\left\{\mathbf{1}\otimes\mathbf{1}+\left(\mathrm{e}^{-\lambda_j t/2}-1\right)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)\right\} \qquad (13)
\end{aligned}
$$

where we have used facts

(a) $\left\{P_j\otimes Q_j^T+Q_j\otimes P_j^T \mid j\in\{j\}\right\}$ are projectors commuting with each other.

(b) $\mathrm{e}^{\lambda R}=\mathbf{1}+\left(\mathrm{e}^{\lambda}-1\right)R$ if R is a projector.

Here we set $R_j=P_j\otimes Q_j^T+Q_j\otimes P_j^T$. For $i<j<k$ we obtain

(c) $R_iR_j=\left(P_i\otimes Q_i^T+Q_i\otimes P_i^T\right)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)=P_i\otimes P_j^T+P_j\otimes P_i^T$.

(d) $R_iR_jR_k=\left(P_i\otimes P_j^T+P_j\otimes P_i^T\right)\left(P_k\otimes Q_k^T+Q_k\otimes P_k^T\right)=0$.

From (13) and (c), (d)

$$
\begin{aligned}
(\sharp) &= \prod_{\{j\}}\left\{\mathbf{1}\otimes\mathbf{1}+\left(\mathrm{e}^{-\lambda_j t/2}-1\right)R_j\right\}\\
&=\mathbf{1}\otimes\mathbf{1}+\sum_j\left(\mathrm{e}^{-\lambda_j t/2}-1\right)R_j+\sum_{j<k}\left(\mathrm{e}^{-\lambda_j t/2}-1\right)\left(\mathrm{e}^{-\lambda_k t/2}-1\right)R_jR_k\\
&=\mathbf{1}\otimes\mathbf{1}+\sum_j\left(\mathrm{e}^{-\lambda_j t/2}-1\right)\left(P_j\otimes Q_j^T+Q_j\otimes P_j^T\right)+\\
&\quad\sum_{j<k}\left(\mathrm{e}^{-\lambda_j t/2}-1\right)\left(\mathrm{e}^{-\lambda_k t/2}-1\right)\left(P_j\otimes P_k^T+P_k\otimes P_j^T\right). \qquad (14)
\end{aligned}
$$

Therefore

$$\widehat{\rho}(t) \approx \left(\mathrm{e}^{-itH} \otimes \left(\mathrm{e}^{itH}\right)^T\right) \left\{ \mathbf{1} \otimes \mathbf{1} + \sum_j \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(P_j \otimes Q_j^T + Q_j \otimes P_j^T\right) + \right.$$
$$\left. \sum_{j<k} \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(\mathrm{e}^{-\lambda_k t/2} - 1\right) \left(P_j \otimes P_k^T + P_k \otimes P_j^T\right) \right\} \widehat{\rho}(0). \tag{15}$$

Coming back to matrix form by use of (7) we finally obtain

$$\rho(t) \approx \mathrm{e}^{-itH} \left\{ \rho(0) + \sum_j \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(P_j \rho(0) Q_j + Q_j \rho(0) P_j\right) + \right.$$
$$\left. \sum_{j<k} \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(\mathrm{e}^{-\lambda_k t/2} - 1\right) \left(P_j \rho(0) P_k + P_k \rho(0) P_j\right) \right\} \mathrm{e}^{itH} \tag{16}$$

or

$$\rho(t) \approx \mathrm{e}^{-itH} \left\{ \rho(0) + \sum_j \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(P_j \rho(0) Q_j + Q_j \rho(0) P_j\right) + \right.$$
$$\left. \frac{1}{2} \sum_{j \neq k} \left(\mathrm{e}^{-\lambda_j t/2} - 1\right) \left(\mathrm{e}^{-\lambda_k t/2} - 1\right) \left(P_j \rho(0) P_k + P_k \rho(0) P_j\right) \right\} \mathrm{e}^{itH} \tag{17}$$

for $j, k \in \{j\}$. This is the main result.

A comment is in order. In the two qubit system a general density matrix is written as

$$\rho(t) = \frac{1}{4} \left(\mathbf{1}_2 \otimes \mathbf{1}_2 + p_i(t) \sigma_i \otimes \mathbf{1}_2 + q_j(t) \mathbf{1}_2 \otimes \sigma_j + r_{ij}(t) \sigma_i \otimes \sigma_j\right)$$

where we have used the Einstein's notation on summation . Using this expression one tries to solve the equation coming from pure decoherence term

$$\frac{\partial}{\partial t} \rho = -\frac{1}{2} \sum_{\{j\}} \lambda_j \left(P_j \rho + \rho P_j - 2 P_j \rho P_j\right) = -\frac{1}{2} \sum_{\{j\}} \lambda_j \left(P_j \rho Q_j + Q_j \rho P_j\right).$$

The equation is then reduced to a set of (relatively simple) equations of $\{p\}$, $\{q\}$ and $\{r\}$. However, such a method (trial) is irrelevant as shown in the paper. Our method is quite general !

In this paper we considered the master equation with the dissipative being a set of projectors and constructed the exact solution under the commutable approximation. This is just the first step for constructing a general solution for the equation.

In order to take one step forward we must take the "interaction" term $I(A,B)$ in (11) into consideration. However, such a method to calculate it has not been known as far as we know. Therefore it may be reasonable to restrict our target to some simple models. Further work will be needed and we will report it in a near future, [6].

On the other hand we are studying some related topics, see [7] and [8]. However, we make no comment on them in the paper.

Lastly, we conclude the paper by stating our motivation. We are studying a quantum computation (computer) based on Cavity QED (see [9] and [10]), so to construct a more realistic model of (robust) quantum computer we have to study a severe problem coming from decoherence. This is our future task.

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